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Quantum Information and Entropy

Subhash Kak[1]

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Thermodynamic entropy is not an entirely satisfactory measure of information of a quantum state. This entropy for an unknown pure state is zero, although repeated measurements on copies of such a pure state do communicate information. In view of this, we propose a new measure for the informational entropy of a quantum state that includes information in the pure states and the thermodynamic entropy. The origin of information is explained in terms of an interplay between unitary and non-unitary evolution. Such complementarity is also at the basis of the so-called interaction-free measurement.

KEY WORDS: Quantum information; quantum entropy; interaction-free measurement; origin of information.

1. INTRODUCTION

Thermodynamic entropy measures the disorder of a system, and although we will show that it is not identical to informational entropy, it is used freely in physics and employed in settings where not only order but also what is intuitively "information" are involved. In the popular view that information is the foundational stuff of reality, what is meant is informational entropy, but what is used is thermodynamic entropy.

Thermodynamic entropy considers the number of structural arrangements associated with the system, whereas informational entropy is about choices made in a communications context. The argument might be made that information is ultimately physical and, therefore, there should be a thermodynamic basis to informational entropy. But this is true only as long as it is possible to characterize the information process in terms of statistical ensembles, which may not be the case in situations relating to communicating agents or in quantum cosmology.

In a classical system, informational entropy may be best viewed in the context of a game between the source, X, and the receiver, Y, in which, upon receipt of signal, the receiver discovers which signal was actually sent (Here we don't

[1] Department of Electrical & Computer Engineering, Louisiana State University, Baton Rouge, Baton Rouge, LA 70803-5901; e-mail: kak@ece.lsu.edu

concern ourselves with complications arising out of noisy communication). The source chooses a signal out of an ensemble, and the choices are repeated in accord with the language (patterns of signals) that connects it with the receiver. For physical systems, the game may be perceived as being played between Nature and the physicist.

The same idea of the game also underlies quantum information (Kak, 2006). But here the situation is more complex, because the quantum state could be pure or mixed, and these two cases are very different from the point of view of measurement. A mixed state is a statistical mixture of component pure states, and its entropy is computed by the von Neumann measure in a manner that is similar to the entropy for classical states. A pure state is completely described by its state function and its von Neumann entropy is zero.

It is important to note the asymmetry between the quantum system and the physicist. From the point of view of the preparer of the states, the pure state carries information that is limited by the "relationship" between the source and the receiver, and by the precision of the receiver's measurement apparatus. The source may choose out of an infinity of possibilities, and the dependence on the "relationship" implies that the pure state's information will vary from one receiver to another.

For the source, the information generated by him equals the probability of choosing the specific state out of the possibilities available to him (this is the states *a priori* probability). If the set of choices is infinite, then the "information" generated by the source is unbounded. On the other hand, due to the probabilistic nature of the reception process, not all the information at the source is obtained at the receiver by his measurement.

In recent years several theories have been advanced that assign finite entropy to matter and space (Bekenstein, 1973). The finite value of entropy for a given volume has been taken to mean that matter cannot be subdivided infinitely, and that the fundamental entity relating to matter is a bit (1 or 0) of information.

However, this approach of discretization hasn't been very successful. Part of the fault may lie in the limitations of the current concept of quantum entropy. In particular, von Neumann entropy is not the right measure in the asymmetric situation where the choice of the state itself carries information.

In this paper, we argue that an unknown quantum pure state, when viewed in the context of the game between the source and the receiver, communicates information just as a mixed state. We propose a measure for informational entropy and show that it may be seen as the sum of information in the pure states and the thermodynamic entropy. The origin of information is seen as a consequence of the interplay between unitary and non-unitary evolution, which makes it possible to transform one type of information into another. The significance of this complementarity is considered for the case of "interaction-free" measurements. This complementarity indicates that a fundamental duality is essential for information, which means that complete unification will not be possible.

2. CLASSICAL AND VON NEUMANN MEASURES OF INFORMATION

Let the source be associated with a random variable, X, that takes values from a discrete set $x_1, x_2, \ldots, x_n$ with probabilities $p(x_1), p(x_2), \ldots, p(x_n)$. The information associated with the receipt of signal x_i is $-\log_2 p(x_i)$. The average information, or Shannon entropy, of the source is:

$$H(X) = -\sum_i p(x_i) \log\ p(x_i). \tag{1}$$

The maximum value of entropy, obtained for the case when all signals are equally likely, is log n. When the variable X is continuous with the probability density $f_X(x)$, its entropy $H(X)$ is given by the expression:

$$H(X) = h(X) - \lim_{\Delta x \to 0} \log\ \Delta x, \tag{2}$$

where $h(X)$ is the Boltzmann or differential entropy:

$$h(X) = \int_{-\infty}^{\infty} f_X(x) \log \left[\frac{1}{f_X(x)} \right] dx, \tag{3}$$

and Δx is the precision associated with the measurement of the variable. The value of $H(X)$ depends on the details of the experimental arrangement. Its maximum value, when the precision is absolute, is infinite.

If it is taken that the measurement has its own uncertainty, then the value of entropy is finite that is given by the well-known information capacity theorem.

The measure of entropy (1), when generalized for a quantum system characterized by the density operator ρ, is the von Neumann entropy:

$$S_n(\rho) = -tr(\rho\ \log\ \rho), \tag{4}$$

This may be equivalently written as:

$$S_n(\rho) = \sum_x \lambda_x \log\ \lambda_x, \tag{5}$$

where λ_x are the eigenvalues of the density matrix ρ associated with the system.

The von Neumann entropy may be viewed as the average information the experimenter obtains in the repeated observations of the very many copies of an identically prepared mixed state. The entropy $S(\rho)$ for the mixed state

$$\rho = \begin{bmatrix} p & 0 \\ 0 & 1-p \end{bmatrix} \tag{6}$$

is equal to

$$-p \log p - (1-p)\log(1-p). \tag{7}$$

The von Neumann entropy of a pure state is zero, indicating that once it has been identified then there is no further information to be obtained from its copies, which is not the case with a mixed state.

3. ENTROPY OF THE UNIVERSE

When applied to matter, some general arguments related to degrees of freedom are invoked to estimate that the entropy of a physical system is equal to

$$S_n \leq \frac{A}{4} \tag{8}$$

where A is the area in Planck units equal to $\hbar G/C^3$. This is the Bekenstein bound (Bekenstein, 1973), given originally in the form $S \leq 2\pi EL$, where L is the linear size of the region, and E is the energy. Gerard 't Hooft later generalized it ('t Hooft, 1993) to the form involving $A/4$ nats (1 bit equals ln 2 nats), and as the holographic principle (Susskind, 1995) it is supposed to apply to all matter. The total quantity of bits in this approach is a measure of the degrees of freedom associated with the system. An informational approach based on fundamental limitation to precision of the measurement also indicates finite entropy. But if such a limitation is not justified, then information associated with space and matter should be infinite.

A physical system is described in terms of its state at some specific time, and the dynamical laws governing its evolution. The idea of entropy tells us which configurations are more likely than others. For dynamical laws, one expects dimensionless parameters in a theory to be of order unity, reflecting the interaction between comparable processes. But the gravitational, weak, and strong forces have characteristic dimensions that are of very different orders of magnitude. Furthermore, the actual range spanned by parameters related to gravitation, electroweak and strong forces, and the Hubble scale characteristic of cosmology is immense, indicating that we may not be looking at the question the correct way.

The distribution of matter on very large scales has been found to be approximately homogeneous and isotropic. The current data is interpreted to mean that distant galaxies are expanding away from each other in accordance with Hubble's law.

By extrapolation into the past, the universe is taken to have originated about 14 billion years ago in a superdense state. If it was a state in thermal equilibrium, then this would mean a violation of the second law of thermodynamics, since the initial state should be in an entropy minimum.

In the current synthesis, "ordinary matter," consisting of particles described by the Standard Model of particle physics, accounts for only about 4% of the total energy of the universe. It is believed that another 23% comes from particles yet to be discovered, or "dark matter," and a further 73% is "dark energy," generated by an unknown force.

The matter in the universe appears to be smoothly distributed, and the deviations from smoothness are taken as a consequence of initial conditions (Penrose, 1989). The entropy of matter and radiation in the observable universe is approximately 10^{88}, where it is assumed that the background radiation entropy for each baryon is 10^8. (The entropy here is in "natural units," in which the Boltzmann's constant is taken to be unity.) Initially this was mostly in the form of radiation, but now it is assumed to be mainly concentrated in the entropy of the black holes at the centers of galaxies.

With probably more than ten billion galaxies with million-solar-mass black holes at their centers, the current entropy in black holes is of the order of 10^{100}. If all the matter in the observable universe were to be combined into one giant black hole, the entropy would be significantly larger 10^{120}.

The estimated entropy of the universe is rather small, given the size of the universe. Although it is believed to be increasing due to the second law of thermodynamics, but it is lagging its potential maximum. Others have argued that the initial entropy must have been still lower, with estimates from 10^{10} to 10^{20}.

The von Neumann measure leads to the puzzle of how information arose in the universe. If we were to assume that the total universe quantum state in the beginning was pure, then the information associated with the universe as a whole was zero. On the other hand, if it is assumed that the deviations from perfect isotropy represented the initial entropy, then the amount of this entropy was rather small. If the components now are entangled states, their ancestor states at the beginning should also have been entangled.

A related puzzle is the emergence of non-unitary evolution in the universe. There can be no information in a universe completely governed by unitary evolution. The resolution to this puzzle is to assume that the physical universe comes with evolution that has unitary as well as non-unitary components. This duality is what makes information possible in the universe. It follows that one cannot assume a single mechanism behind the two evolutions.

The question of how information is increasing is a central one in physics. Our proposed measure provides a resolution by showing how pure states carry entropy.

4. INFORMATIONAL ENTROPY, $S_i(\rho)$, OF A QUANTUM SYSTEM

Unlike a classical state that is completely known when it is measured, the process of measurement of a quantum state merely determines its projection along chosen basis vectors, and this projection is probabilistic.

Once a classical variable has been measured (examples being location or mass), it is correct to assume that further measurements will not provide any new information. In the case of location variable, we know that the object will continue at its position owing to the fact that the object can, in principle, be isolated from the environment. Likewise, the mass values, in further measurements, will be identical to the first measurement.

Let the setting for the game related to quantum information be one where the source is producing identical copies of an elementary quantum state. In contrast to the classical case, there are two different situations that one must consider. In general, one doesn't know whether the state is pure or mixed. The game for the receiver is to determine this state as closely as possible, after examining as many copies of the state as is required. The entropy then is the average information communicated about the unknown state at any point in the measurement process.

It is assumed that the source and the receiver use the same basis vectors for the representation and the measurement of the states. This assumption is necessary to establish the baseline of the game between the source and the receiver.

For the mixed state, the entropy is reasonably given by the von Neumann value. As mentioned before, the von Neumann entropy for a pure state is zero. But an unknown pure state will communicate real information to the receiver, indicating that the von Neumann entropy is not a reasonable measure in this case.

We propose that S_i represent the informational entropy of the quantum system with the density matrix ρ:

$$S_i(\rho) = -\sum_i \rho_{ii} \log \rho_{ii}. \tag{9}$$

This represents the average uncertainty that the receiver has in relation to the quantum state *for each measurement*. Should the manner of the preparation of the pure state be known to the observer, he can choose a basis state function that would completely describe it, and there would indeed be no information associated with it.

By appropriately adjusting the basis vectors, the receiver can change the value of this entropy. The value of $S_i(\rho)$ is not a measure of the entropy at the transmitting end. It is the amount of entropy of the quantum system that is accessible to the receiver.

Some properties of S_i are:

1. $S_i(\rho) \geq S_n(\rho)$, and the two are equal only when the density matrix has only diagonal terms.
2. $S_n(\rho)$ is obtained by minimizing $S_i(\rho)$ with respect to all possible unitary transformations. In other words,

$$S_n(\rho) = \inf_U S_i(U_\rho U^\dagger) \tag{10}$$

3. The maximum value of S_i is infinity, true for the case where the number of components is infinite.

From the point of view of the source, a finite system can also carry infinite information. Let us now, for convenience, assume that the quantum state is coded in the polarization of photons:

$$|\phi\rangle = \alpha|0\rangle + \beta|1\rangle, \tag{11}$$

where the states $|0\rangle$ and $|1\rangle$ represent horizontally and vertically polarized photons, respectively, and α is real. The information exchange protocol may be defined by the transmission, according to a clock, of photons, which are detected using appropriate circuits and polarizing filters. The task of the receiver is to estimate the value of α (and, implicitly, β). The value of α could be written down as a decimal sequence in a string of 0 s and 1 s, that represents a secret.

As far as the receiver is concerned, only one bit of information is obtained from any single photon. On the other hand, since a large number of identically prepared photons is available, one could hope to find the exact probability amplitude values α and β by testing out different hypotheses related to the nature of the state function. To determine these values with any precision, testing of a large number of the photons is required so as to approach ever closer the true, unknown value.

The measurement could use a transformation, so that the transformed photon is rotated to the $|0\rangle$ state. Thus, the receiver needs a procedure where the measured values would let him find the transformation matrix:

$$G = \begin{bmatrix} \alpha & \beta^* \\ -\beta & \alpha \end{bmatrix}. \tag{12}$$

If each test is assumed to provide one bit of information, then such a specifically prepared photon carries information determined by the precision available to the receiver to distinguish between different component states.

For further simplicity, we might consider the qubit to be defined such that both the values of α and β are real, and $|\phi\rangle = \cos\theta|0\rangle + \sin\theta|1\rangle$. We can speak of the unknown state to be associated with the angle θ as follows:

$$G^+ = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix} \tag{13}$$

It appears that there is no efficient deterministic algorithm to estimate G^+.

Conjecture: There is no deterministic algorithm that will identify G^+ in $O(n^k)$ steps, where n is the number of quantization levels of θ that can be distinguished by the receiver.

At worst this problem belongs to the **NP** class, because if an oracle were to guess the correct G^+, it is easy to check it, since applying this transformation the photons will be transformed to the state $|0\rangle$.

A Cryptographic Context

The above scenario may be viewed in the context of cryptography as follows. Alice and Bob agree to use a n-qubit long sequence of photons with varying polarization angles that represents their shared signature, which is unknown to the eavesdropper. The sent message can be signed by each with this unique signature sequence that follows the data sequence, and since the recipient knows what to expect, it can be validated.

Since the eavesdropper, Eve, cannot use actual polarization angles (the probability of getting that correct being infinitesimally small), she can match the projections of the signature bits with her own guessed sequence of 0 s and 1s. She has a probability of 2^{-n} of guessing the projection of the sequence along specific basis vectors.

Note that even if her guessed sequence turned out to be correct, it is unlikely to work at future times, since the polarization angles associated with the qubit sequence are unknown to her, and their projection to any basis states chosen by her are going to vary from trial to trial.

Clearly the information associated with each qubit in this setting is infinite. It is incorrect, therefore, to assign finite entropy to a system in the case of maximal ignorance even if the system is finite.

5. PROPERTIES OF INFORMATIONAL ENTROPY

Consider that the quantum system is represented by the density operator ρ, which is an ensemble of pure states $|\phi_i\rangle$ with probabilities p_i and a mixed state with density operator ρ_o with probability p_o in the following manner:

$$\rho = \sum_i p_i |\phi_i\rangle\langle\phi_i| + p_o \rho_o. \tag{14}$$

The total informational entropy of the system will be given by:

$$S_i(\rho) \geq \sum_i p_i S_p(\phi_i) + p_o S_n(\rho_o) \tag{15}$$

where $S_p(\phi)$ represents the entropy of the pure state $|\phi\rangle = \sum_k c_k |a_k\rangle$:

$$S_p(\phi) = -\sum_k |c_k|^2 \log |c_k|^2 \tag{16}$$

that is a companion to the mixed state. The reason why the left hand side can be larger than the sum of the individual parts is that if the pure components are chosen inappropriately, as aligned with the basis components at the receiver, one would obtain no contribution towards entropy from such components.

Example 1. Let the system density operator be described by:

$$\rho = \begin{bmatrix} .5 & .25 \\ .25 & .5 \end{bmatrix}. \tag{17}$$

Here,

$$\rho = 0.5 \times \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} + 0.5 \times \begin{bmatrix} .5 & 0 \\ 0 & .5 \end{bmatrix}. \tag{18}$$

As far as the receiver is concerned, there is no way for him to know *a priori* whether the quantum state received is pure or mixed. In each test of the very many copies of the state available to him (assumed in our communication protocol), he receives one bit of information. The informational entropy in the beginning is 1 bit.

A simple calculation tells us that $S_i(\rho) = 1$ bit, whereas $S_n(\rho) = 0.811$ bit. On the other hand, S_p is 1 bit, and $S_i = 0.5 \times 1 + 0.5 \times 1 = 1$ bit. Clearly, informational entropy is a better measure than the von Neumann measure in this situation.

Example 2. Let the system density operator be described by:

$$\rho = \begin{bmatrix} .71 & .15 \\ .15 & .29 \end{bmatrix}. \tag{19}$$

The informational entropy for this example is $-.71 \log .71 - .29 \log .29 = 0.868$ bits. But this quantum state may be written down as the statistical ensemble:

$$\rho = 0.3 \times \begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix} + 0.7 \times \begin{bmatrix} .8 & 0 \\ 0 & .2 \end{bmatrix}. \tag{20}$$

The first part of the ensemble represents a pure state with a probability of 0.3 and the second part is a mixed state with a probability of 0.7. One can easily calculate that the individual components have information of 1 bit and 0.722 bits, respectively. The sum of the two entropies is therefore:

$$0.3 \times 1 + 0.7 \times .722 = 0.805 \tag{21}$$

bits, which is less than the informational entropy.

Example 3. Let the system density operator be described by:

$$\rho = \begin{bmatrix} .75 & 0 \\ 0 & .25 \end{bmatrix} \tag{22}$$

The value of $S_i(\rho) = 0.559$ bits.

This system can also be expressed as a statistical ensemble with two pure state components:

$$|a\rangle = \sqrt{\frac{3}{4}}|0\rangle + \sqrt{\frac{1}{4}}|1\rangle \tag{23}$$

$$|b\rangle = \sqrt{\frac{3}{4}}|0\rangle - \sqrt{\frac{1}{4}}|1\rangle \tag{24}$$

and

$$\rho = \frac{1}{2}|a\rangle\langle a| + \frac{1}{2}|b\rangle\langle b|. \tag{25}$$

The computation of the total entropy for this case is then:

$$S_i = \frac{1}{2}S_i(a) + \frac{1}{2}S_i(b) \tag{26}$$

Using the value of S_i for each of the components, we get:

$$S_i = \frac{1}{2} \times 0.559 + \frac{1}{2} \times 0.559 = 0.559. \tag{27}$$

This is exactly equal to the earlier calculation. Or using two different ensembles of quantum states corresponding to the same density matrix gives us identical results upon the use of the informational entropy measure S_i.

6. THE ORIGIN OF INFORMATION

Suppose the universe initially was in a pure or a low-entropy quantum state, how did high entropy states arise? A part of the increase of entropy is due to the second law of thermodynamics, but this contributes a small share to the overall value. Likewise, the expansion of the universe will contribute to the increase, but this also does not square up with the actual increase that has occurred.

Quantum evolution of a pure state leaves it unchanged and, therefore, that cannot be the explanation for it.

But if we were to consider many particles in a pure state, say $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, then their sequential observation will create mixed states of non-zero von Neumann entropy.

Given the fact that we have both unitary, U, and non-unitary, M_i, or measurement, operators, the density operator for each elementary state will change either to:

$$|\phi\rangle_{\text{new}} = \begin{cases} U|\phi\rangle & \text{unitary evolution} \\ \frac{M_i|\phi\rangle}{\sqrt{\langle\phi|M_i^{\dagger}M_i|\phi\rangle}} & \text{non-unitary evolution} \end{cases} \tag{28}$$

When only non-unitary operators are used for the evolution, the elementary state will change from the pure state $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$ to the mixed state given by the density matrix:

$$\rho = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}. \tag{29}$$

Its informational entropy would then have transformed completely from that of the pure state to that of the mixed state, and its von Neumann entropy would now be finite.

The existence of non-unitary operators requires the presence of low-entropy structures that in themselves could not have arisen in a universe governed by a single law. If gravitation is viewed as the force that causes matter to aggregate, making non-unitary evolution possible, then gravitation and quantum theory would for ever be irreconcilable.

This view of the problem of how information increases is to postulate non-unitary evolution as a part of the earliest universe (Kak, 2001), suggesting that unification has its limits.

7. COMPLEMENTARITY AND INTERACTION-FREE MEASUREMENT

We now consider information in the framework of distinguishing between two states of an experimental arrangement that has traditionally been associated with *interaction-free measurement* (IFM). Our intention is to check the usefulness of the informational entropy measures in this situation and to show that complementarity provides the most reasonable explanation.

There are several versions of IFM, which are basically variants of the Young's double-slit experiment, although for convenience the setting is the Mach-Zehnder interferometer (Fig. 1). The basic idea of each is to focus on the counter-intuitive fact that when the experiment is so set up that it is possible to determine which path the photon took, the photons exhibit particle-like behavior, and if it is not possible to do so, then they exhibit wave-like behavior.

In Fig. 1, a photon (from a source of single photons) reaches the first half-silvered mirror, A (beam splitter), which has a transmission coefficient 1. The transmitted and reflected parts of the photon wave reunite at another, similar half-silvered mirror at D. The beam splitters and fully-silvered mirrors (B and C) are arranged in such a way that the photon is always detected by D_1, and never detected

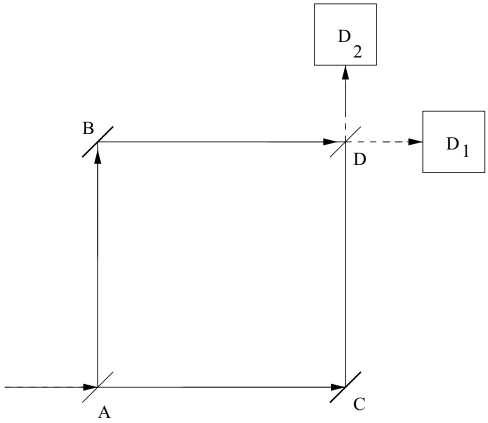

Fig. 1. The Mach-Zehnder interferometer.

by D_2. This corresponds to the baseline case where the entropy is zero, which is reasonable given that there is no uncertainty associated with the process.

The IFM setting is associated with a modification to the Mach-Zehnder interferometer by the use of a springy mirror C^+ in place of C as in Fig. 2. When the mirror C is rigid, the photons will exhibit wave nature; when the mirror is not rigid, the photons will exhibit particle nature. Figures 1 and 2 represent pure and mixed states, respectively.

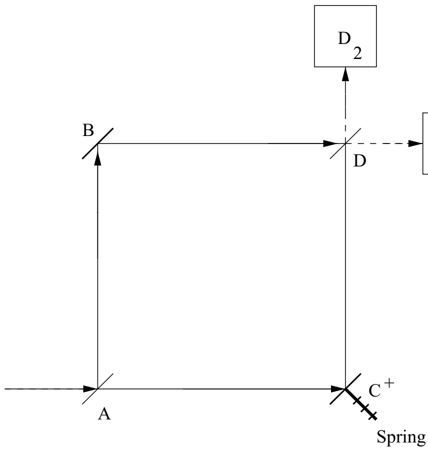

Fig. 2. Mach-Zehnder interferometer with the mirror C^+ attached to a spring.

There are three possible outcomes of this system:

1. photon absorbed, probability 1/2
2. detector D_1 clicks, probability 1/4
3. detector D_2 clicks, probability 1/4.

The difference between the two cases of Figs. 1 and 2 is that of the difference between a known pure state and a mixed state with probabilities that are one-half in each of the component states. It is appropriate to get the entropy in terms of the clicks.

In Fig. 2, there is an equal probability that the lower or the upper paths will be chosen. The choice of the lower path leads to the absorption of the photon, whereas the choice of the upper path leads to equal probability that it will end up in D_1 or D_2. If detector D_2 clicks, one can claim that the photon was "aware" that the mirror C was not rigid and it took the upper path, and the spring in C^+ did not have to respond to the photon. This arrangement corresponds to an entropy of 1.5 bits.

The claim is (Elitzur and Vaidman, 1993) that the method makes it possible to sense an infinitely sensitive mirror without interacting it with a probability of 1/4. In reality, no measurement was necessary since we already knew that the mirror C^+ is springy.

If we wish to obtain information, it is essential that there be alternatives. If we take it that we don't know if the experimental arrangement consists of C or C^+ (both of which occur with equal probability), then the outcomes are:

1. photon absorbed, probability 1/4
2. detector D_1 clicks, probability 5/8
3. detector D_2 clicks, probability 1/8.

Since this case is a mixture of the previous two cases, the entropy will should be intermediate to the previous values. A simple calculation gives the value as 1.299 bits.

Using Bayes' theorem, we know that

$$p(C^+|D_2) = \frac{p(D_2|C^+)p(C^+)}{p(D_2)} = 1 \tag{30}$$

$$p(C^+|D_1) = \frac{p(D_1|C^+)p(C^+)}{p(D_1)} = 1/5. \tag{31}$$

It is clear that it is the geometry of the experimental arrangement that maps to different probabilities as listed above. The alternatives of C and C^+ lead to pure and mixed states, but the entropy associated with each of them is the same.

In the Many Worlds Interpretation (MWI), which is favoured by those who accept the reality of "interaction-free measurement," one assumes several worlds existing at the same time, and in each world we perceive what is a small part of what is in the universe. The laws of physics relate to the whole universe, but viewed in the partial description of any specific world, one may have paradoxical situations such as that of measurement without interaction. In the framework of the MWI we find the springy mirror because in another world it was indeed examined by a photon.

In the Complementarity Interpretation, one must speak of the entire experimental arrangement. The arrangement guarantees that the measurement is made, albeit indirectly.

For example, the placement of C required prior measurement. If the measurement consists of choosing between the arrangements C and C^+, the placement of C^+ is associated with an uncertainty due to the fact that one doesn't know in advance whether the mirror will absorb the photon or not. Since

$$\Delta x \Delta p \geq \hbar/2, \tag{32}$$

and $\Delta p = \hbar/\lambda$, therefore,

$$\Delta x \geq \lambda/4\pi, \tag{33}$$

where λ is the wavelength associated with the photon. If the location of C^+ cannot be precise, there will correspondingly be uncertainty in the ability to distinguish between C and C^+.

8. NON-UNITARY EVOLUTION

Although non-unitarity is complementary to unitarity, it may be seen as being generated by the measurement process alone. Let us consider the case where the state is evolving with time. Let $|\psi_o\rangle$ be the initial state of quantum system, and let the state evolve into $|\psi_t\rangle$ in time t. Let the Hamiltonian characterizing the evolution be time-independent.

$$|\psi_t\rangle = \exp\left(-\frac{i}{\hbar}Ht|\psi_o\right). \tag{34}$$

Because of the continuing evolution of the state, any entropy computation based on the von Neumann or the proposed pure state entropy measure will be of fleeting significance. It appears, therefore, that the entropy should be related to the unknown Hamiltonian H.

A measure of this entropy would be the frequency with which one needs to observe the system so as to freeze the state, which brings us to the so-called Zeno effect (Misra and Sudarshan, 1977). We can represent the evolution of the state by the following approximation:

$$|\psi_t\rangle \approx \left(1 - \frac{i}{\hbar}Ht - \frac{1}{2\,\hbar^2}H^2t^2\right)|\psi_o\rangle. \tag{35}$$

The correlation between the states at time 0 and t is:

$$\langle\psi_o|\psi_t\rangle \approx \langle\psi_o|\psi_o\rangle - \frac{it}{\hbar}\langle\psi_o|H|\psi_o\rangle - \frac{t^2}{2\hbar^2}\langle\psi_o|H^2|\psi_o\rangle$$

$$= 1 - \frac{it}{\hbar}\langle\psi_o|H|\psi_o\rangle - \frac{t^2}{2\hbar^2}\langle\psi_o|H^2|\psi_o\rangle \tag{36}$$

$$|\langle\psi_o|\psi_t\rangle|^2 \approx \left(1 - \frac{t^2}{2\hbar^2}\langle\psi_o|H^2|\psi_o\rangle\right)^2 + \frac{t^2}{\hbar^2}\langle\psi_o|H|\psi_o\rangle^2$$

$$= 1 - \frac{t^2}{\hbar^2}\langle\psi_o|H^2|\psi_o\rangle + \frac{t^2}{\hbar^2}\langle\psi_o|H|\psi_o\rangle^2 \tag{37}$$

Let $(\Delta E)^2 = \langle\psi_o|H^2|\psi_o\rangle - \langle\psi_o|H|\psi_o\rangle^2$, then

$$|\langle\psi_o|\psi_t\rangle|^2 \approx 1 - \frac{(\Delta E)^2}{\hbar^2}t^2. \tag{38}$$

The evolution suppressing, Zeno case corresponds to repeated observations at times t/n:

$$|\langle\psi_o|\psi_t\rangle|^2 \approx \left(1 - \frac{(\Delta E)^2}{\hbar^2}\frac{t^2}{n^2}\right)^n$$

$$= 1 - \frac{(\Delta E)^2}{\hbar^2}\frac{t^2}{n}. \tag{39}$$

We know that as the number of observations becomes infinite, the state at time t is the same as the state at time $t = 0$:

$$\lim_{n\to\infty}\langle\psi_o|\psi_t\rangle|^2 \approx 1. \tag{40}$$

In our case, the measure of entropy would be the value of n that allows us to freeze the state within the precision available to the receiver.

Consider a photon that is horizontally polarized, which we represent by $|0\rangle$. We can, by using a polarizing filter, oriented in the direction 45°, make half the number of photons collapse to the state $\frac{1}{2}(|0\rangle + |1\rangle)$. In two such observations, the photon's polarization would be steered to 90° with a probability of $\frac{1}{4}$.

If the rotation in each step is $\theta°$, one would need a total of $\frac{\pi}{2\theta} = n$ steps to rotate the original state of $|0\rangle$ to the state of $|1\rangle$, and this will happen with the probability of

$$(\cos^2\theta)^{\frac{\pi}{2\theta}}. \tag{41}$$

Figure 3 illustrates this and the probability of steering the photon to the desired final state of $|1\rangle$ become quite close to 1 as n approaches 100. For $n = 90$, the probability is 0.973.

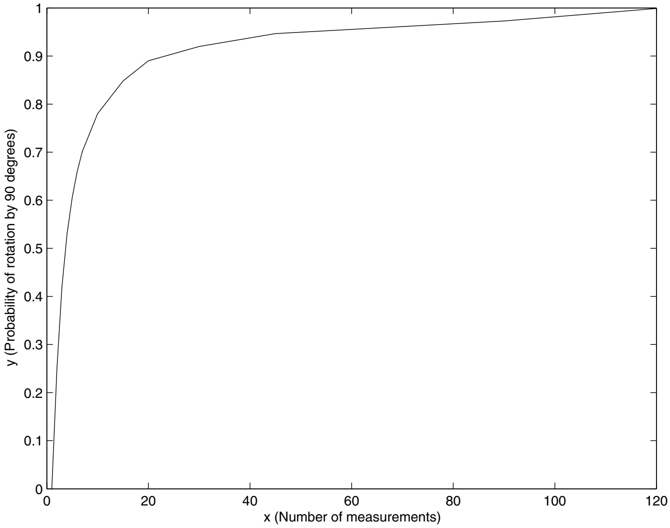

Fig. 3. Observation driven evolution.

For someone who did not know that the photon was being steered by repeated measurements, the evolution of the photon would be viewed as a consequence of the Hamiltonian associated with the system. If the measurements are made at regular intervals, the Hamiltonian would be considered time-dependent. The rotation would be largest at the first step and it will progressively decrease with each new step. Alternatively, one may view the rotation process to be faster (associated with larger energy) at first with the speed tapering off as observations continue.

If it is valid to see non-unitarity as resulting from wave collapse alone, then the search for hidden variable theories of quantum mechanics will be futile. In this view, complete unification is not possible.

9. CONCLUDING REMARKS

Considering the information transfer problem from the point of view of the preparer of the state and the experimenter, it is clear that both mixed and pure states provide information to the experimenter. For a two-component elementary mixed state, the most information in each measurement is one bit, and each further measurement of identically prepared states will also be one bit.

For an unknown pure state, the information in it represents the choice the source has made out of the infinity of choices related to the values of the probability amplitudes with respect to the basis components of the receiver's measurement apparatus. The maximum information in a pure state is thus infinite. On the other hand, each measurement of a two-component pure state can provide one bit of information. But if it is assumed that the source has made available an unlimited number of identically prepared states, the receiver can obtain additional information from each measurement until the probability amplitudes have been correctly estimated. Once that has occurred, unlike the case of a mixed state, no further information will be obtained from testing additional copies of this pure state.

The receiver can do this by adjusting the basis vectors so that he gets *closer* to the unknown pure state. As the adjustment proceeds, the amount of information that he would obtain from each measurement will decrease. The information that can be obtained from such a state in repeated experiments is potentially infinite in the most general case.

But if the observer is told what the pure state is, the information associated with the states vanishes, suggesting that a fundamental divide exists between objective and subjective information.

The analysis of this paper is consistent with the positivist view that one cannot speak of information associated with a system excepting in relation to an experimental arrangement together with the protocol for measurement. The experimental arrangement is thus integral to the amount of information that can be obtained.

The informational measure proposed in this paper resolves the puzzle of entropy increase in the universe. We can suppose that the universe had immensely large informational entropy in the beginning, a portion of which has, during the physical evolution of the universe, transformed into thermodynamic entropy. If we take it that the dichotomy of quantum processes and gravitation is responsible for unitary and non-unitary evolution, then it should not be possible to unify the two.

The process of scientific discovery in terms of the knowledge of its laws may be taken to be the unveiling of the basis vectors of the pure state associated with the bulk of informational entropy. This process will be unending, even as we come ever closer to the essential bases.

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